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ON A STOCHASTIC CONTROL PROBLEM WITH EXIT CONSTRAINTS+

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ON A STOCHASTIC CONTROL PROBLEM WITH EXIT CONSTRAINTS

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ABSTRACT: In this paper the logarithmic transformation of Fleming [1] is used to discuss a specific problem of controlled diffusions. The problem is to minimize a certain quadratic functional of the applied drift while satisfying the requirement that the place where the process exits a domain is not in a specified subset of its boundary. The main result is that the solution of this problem is given by the logarithm of a related exit probability.

I. INTRODUCTION AND STATEMENT OF THE THEOREM

In this paper we are going to use the logarithmic transformation discussed in [1] to solve a certain stochastic control problem. To see our problem in perspective, it is convenient to start with the basic result concerning the logarithmic transformation that Fleming discussed in Section 2 of [1].

Let $\Delta \subseteq \mathbb{R}^d$ be a bounded domain with C^2 boundary. The functions $\sigma(x)$ ($d \times d$ non-singular matrix valued) and $b(x)$ ($\mathbb{R}^d$ valued) are assumed to be Lipschitz on $\mathbb{R}^d$ and, along with $\sigma^{-1}(x)$, are bounded. The Markov diffusion process $\xi(t)$ is defined by the stochastic differential equation

$$d\xi(t) = b(\xi(t))dt + \sigma(\xi(t))d\omega$$

where ω is a d-dimensional Brownian motion. τ_D will denote the first exit time of $\xi(t)$ (and also of $\eta(t)$ below) from an open set $D \subseteq \mathbb{R}^d$. $a(x)$ is defined to be $\sigma(x)\sigma(x)'$, where the $'$ denotes transpose. Fleming showed that if $\Phi \in C^2(\mathbb{R}^d)$ and we define

$$g_\Phi(x) = E_x[\exp(-\Phi(\xi(\tau_\Delta)))] \tag{1.1}$$

$$L(x,v) = \frac{1}{2}(b(x)-v)'a(x)^{-1}(b(x)-v),$$

then

$$I_\Phi(x) = -\log g_\Phi(x)$$

is the solution of the following stochastic control problem:

$$\min E\left[\int_0^{\tau_\Delta} L(\eta(s),v(s))ds + \Phi(\eta(\tau_\Delta))\right] \tag{1.2}$$

subject to

$$d\eta(t) = v(t)dt + \sigma(\eta(t))d\omega, \quad \eta(0) = x \tag{1.3}$$

where the minimum runs over all $\eta(t)$ which solve such an equation with $v(t)$ bounded and progressively measurable. Moreover the minimum is achieved by using the feedback control law,

$$v^*(x) = b(x) - a(x)\nabla I_\Phi(x).$$

Now suppose we try $\Phi = +\infty\cdot\chi_{N^c}$ in the above, where $N \subseteq \partial\Delta$. The natural interpretation of (1.1) is

$$g(x) = P_x[\xi(\tau_\Delta) \in N], \tag{1.4}$$

$$I(x) = -\log g(x).$$

In (1.2), Φ becomes an infinite penalty for $\eta(\tau_\Delta) \in N^c$. We interpret this as adding the constraint $\eta(\tau_\Delta) \in N$ to (1.3) and replacing (1.2) by

$$\min E\left[\int_0^{\tau_\Delta} L(\eta(s),v(s))ds\right]. \tag{1.5}$$

Our goal is to prove that $I(x)$ defined by (1.4) is still the solution of this constrained control problem and that $v^*(x)$ given by the same formula is an optimal feedback control. The result that we will prove is stated more carefully below, followed by some remarks preparatory to the proof. The proof is given in Section 2. We conclude with a simple example in Section 3.

The constraint $\eta(\tau_\Delta) \in N$ forces us to consider $v(\cdot)$'s which are unbounded; if $v(\cdot)$ is bounded in (1.3) then $\eta(\tau_\Delta)$ has positive probability of being in any specified open subset of $\partial\Delta$. To describe the collection V_{x_0} of $v(\cdot)$'s to be considered in the minimization in (1.5), we begin by requiring that, associated $v \in V_{x_0}$, there be an increasing family $\{\mathcal{F}_t\}$ of σ-algebras and an adapted d-dimensional Brownian motion ω so that $v(\cdot)$ is progressively measurable with respect to the $\mathcal{F}_t$. Next, we require that a progressively measurable process $\eta^v(t)$ be defined satisfying

(a) $\eta^v(0) = x_0$, (1.6a)

(b) if D is open, contains x_0 and $\bar{D} \subseteq \Delta$, then τ_D (the exit time of η from D) is finite a.s., (1.6b)

(c) for $t < \tau_\Delta$ the following equation is satisfied: (1.6c)

$$d\eta^v(t) = v(t)dt + \sigma(\eta^v(t))d\omega.$$

Here τ_Δ is taken to be $\lim \tau_{D_n}$ where D_n is a sequence of subdomains for which $\bar{D}_n \subseteq D_{n+1}$ and $\cup D_n = \Delta$. (The notation $D_n \uparrow \Delta$ will be used to describe such a sequence.) This definition of τ_Δ does not depend on the particular such sequence used. Implicit in

(c) is the assumption that, for any subdomain D as in (b),

$$\int_0^{\tau_D} |v(t)|dt < \infty \quad \text{a.s.}$$

This is sufficient for us to apply Itô's lemma for $t \le \tau_D$; see [4]. However, $v(t)$ may behave badly as $t \to \tau_\Delta$, consequently $\lim_{t \to \tau_\Delta} \eta^v(t)$ may not exist. We define the statement "$\eta^v(\tau_\Delta) \notin N$" to mean that there exists $t_n \uparrow \tau_\Delta$ and $x \notin N$ so that $\lim \eta^v(t_n) = x$, i.e. there exists a limit point of $\eta^v(t)$, as $t \uparrow \tau_\Delta$, which is not in N. The following <u>admissibility condition</u> is now the precise statement of the constraint mentioned previously:

$$P[\eta^v(\tau_\Delta) \notin N] = 0.$$

V_{x_0} is the collection of all those v for which the above requirements, including the admissibility condition, are all satisfied.

<u>Theorem</u>: Let $N \subseteq \partial\Delta$ be closed and $x_0 \in \Delta$.

(a) Suppose that $v \in V_{x_0}$. Then

$$E[\int_0^{\tau_\Delta} L(\eta^v(s), v(s))ds] \ge I(x_0) \tag{1.7}$$

(b) There exists $v \in V_{x_0}$ for which $v(t) = v^*(\eta^v(t))$, where

$$v^*(x) = b(x) - a(x)\nabla I(x),$$

and equality is achieved in (1.7).

We will use N^0 and N^c to denote the interior and compliment of N computed relative to $\partial\Delta$.

Right away we can dispense with some trivial cases. If $g(x) \equiv 1$, then $I(x) \equiv 0$ and the theorem is trivial. (By the strong maximum principle applied to equation (1.8) below, $g(x)$ is either identically 1 or strictly less than 1. Likewise it is either identically 0 or strictly positive.) If $g(x) \equiv 0$, let N_n be a sequence of closed sets with non-empty interiors so that $\cap N_n = N$. Because the interiors are non-empty,

$$g_n(x) = P_x[\xi(\tau_\Delta) \in N_n] > 0,$$

and $g_n(x) \to 0$. If $\eta^v(t)$ satisfies the admissibility condition for N, then it also does for N_n. The theorem in this case implies that

$$E[\int_0^{\tau_\Delta} L(\eta^v(s),v(s))ds] \geq -\log g_n(x_0).$$

Letting $n \to \infty$ we see that

$$E[\int_0^{\tau_\Delta} L(\eta^v(s),v(s))ds] = +\infty.$$

This shows that it is enough to prove the theorem in the case $0 < g(x) < 1$.

The conditions on Δ, σ and b imply that $g \in C^2(\Delta)$ and satisfies the following equation on Δ:

$$\frac{1}{2} \sum_{i,j} a_{ij} g_{x_i x_j} + \sum_i b_i g_{x_i} = 0 \tag{1.8}$$

with

$$g(x) \to \begin{cases} 1 & \text{on } N^0 \\ 0 & \text{on } N^c. \end{cases}$$

(This can be proved by approximating χ_N by C^2 functions to obtain C^2 approximations of g which satisfy (1.8). Now the Schauder interior estimate [3] Theorem 6.2 gives the precompactness of these approximations and their derivatives up to second order. From this it follows that $g \in C^2(\Delta)$ and satisfies (1.8).) This implies, since $g > 0$ in Δ, that $I \in C^2(\Delta)$ and has boundary behavior

$$I(x) \to \begin{cases} 0 & \text{on } N^0 \\ +\infty & \text{on } N^c. \end{cases} \tag{1.9}$$

One checks, using (1.8), that I must obey

$$0 = \frac{1}{2} \sum_{i,j} a_{ij} I_{x_i, x_j} + H(x, \nabla I(x)) \quad \text{in } \Delta. \tag{1.10}$$

Here, as in [1],

$$H(x,p) = -\frac{1}{2}p'a(x)p + b(x)\cdot p$$

$$= \min_v\{v\cdot p + L(x,v)\}$$

$$= (v^*)\cdot p + L(x,v^*)$$

where $v^* = b(x) - a(x)p$ is the unique v achieving the minimum. Rewriting (1.10) as

$$0 = \frac{1}{2}\sum_{i,j} a_{ij}I_{x_i x_j} + \min_v\{v\cdot I_x + L(x,v)\}, \tag{1.11}$$

we see why one might expect the theorem to be true; (1.11) is the appropriate dynamic programming equation.

II. PROOF OF THE THEOREM

Consider part (b) of the Theorem first. Since v^* is locally Lipschitz, the equation

$$d\eta^*(t) = v^*(\eta^*(t))dt + \sigma(\eta^*(t))d\omega, \qquad \eta^*(0) = x_0$$

has a unique solution for $t < \tau_\Delta$. (See [2] or [4]). Following the standard procedure in such matters, we can take any subdomain D with $\overline{D} \subseteq \Delta$ and apply Itô's lemma in conjunction with (1.11) and the fact that the minimum is achieved by v^* to see that

$$I(x_0) = E[\int_0^{\tau_D \wedge T} L(\eta^*(s), v^*(\eta^*(s)))ds] + E[I(\eta^*(\tau_D \wedge T)]. \qquad (2.1)$$

Now letting $D \uparrow \Delta$ and $T \uparrow +\infty$, the first term on the right approaches

$$E[\int_0^{\tau_\Delta} L(\eta^*(s), v^*(\eta^*(s)))ds]$$

by the monotone convergence theorem. The second term on the right in (2.1) is not so simple; even if we already knew that η^* satisfied the admissibility requirement, since I is unbounded near portions of the boundary, it is conceivable that we get a positive limit for this term. We can, however, draw the following conclusions:

$$I(x_0) \geq E[\int_0^{\tau_\Delta} L(\eta^*(s), v^*(\eta^*(s)))ds] \qquad (2.2)$$

$$I(x_0) \geq E[I(\eta^*(\tau_D \wedge T))]. \qquad (2.3)$$

The second of these implies that $\eta^*(t)$ satisfies the admissibility condition. To prove this let $G_n = \{I(x) < n\}$ and Δ_m be a sequence of subdomains with $\Delta_m \uparrow \Delta$. Set $D_{n,m} = G_n \cap \Delta_m$; it follows that $\tau_{D_{n,m}} = \tau_{G_n} \wedge \tau_{\Delta_m}$. By (2.3),

$$I(x_0) \geq E[I(\eta^*(\tau_{G_n} \wedge \tau_{\Delta_m} \wedge T))]. \tag{2.4}$$

Since $\tau_\Delta = \lim_m \tau_{\Delta_m}$, we see that

$$\sup_{0 \leq s < \tau_\Delta \wedge T} I(\eta^*(s)) > n$$

implies that, for all sufficiently large m,

$$\sup_{0 \leq s < \tau_{\Delta_m} \wedge T} I(\eta^*(s)) \geq n.$$

This in turn, implies that $I(\eta^*(\tau_{G_n} \wedge \tau_{\Delta_m} \wedge T)) = n$. Using this in (2.4) we have that, for fixed n,

$$\begin{aligned} n \cdot P[&\sup_{0 \leq s < \tau_\Delta \wedge T} I(\eta^*(s)) > n] \\ &\leq \lim_{m \to \infty} n \cdot P[\sup_{0 \leq s < \tau_{\Delta_m} \wedge T} I(\eta^*(s)) \geq n] \\ &\leq \lim_{m \to \infty} E[I(\eta^*(\tau_{G_n} \wedge \tau_{\Delta_m} \wedge T))] \leq I(x_0). \end{aligned}$$

Thus,

$$P[\sup_{0\le s<\tau_\Delta} I(\eta^*(s)) = +\infty] = 0.$$

But if $\eta(\tau_\Delta) \notin N$ in the sense described, then by (1.9), $\sup_{0\le s<\tau_\Delta} I(\eta^*(s)) = +\infty$. This proves the admissibility condition.

The property that $\tau_D < \infty$ a.s. for $\bar{D} \subseteq \Delta$ follows from the fact that v^* is bounded on D.

In light of (2.2), the proof of (b) will be complete once we show

$$I(x_0) \le E[\int_0^{\tau_\Delta} L(\eta^*(s),v^*(s))ds].$$

But this is precisely the conclusion of part (a) of the theorem, to whose proof we now turn.

Here we start with the solution of the equation

$$d\eta^v(t) = v(t)dt + \sigma(\eta^v(t))dw; \quad \eta(0) = x_0$$

associated with a given $v \in V_{x_0}$. The objective is to prove

$$E[\int_0^{\tau_\Delta} L(\eta^v(s),v(s))ds] \ge I(x_0).$$

We assume that the quantity on the left is finite, for otherwise there is nothing to prove. The usual Itô calculation has the same difficulty as before; how do we know that $E[I(\eta^v(\tau_D \wedge T))] \to 0$ as as $D \uparrow \Delta$ and $T \uparrow \infty$? This difficulty can be circumvented by replacing the boundary values $+\infty\chi_{N^c}$ for $I(x)$ with some bounded approximations Φ_n. Specifically, take a sequence $\Phi_n \in C^2(\mathbb{R}^d)$ with $0 \leq \Phi_n \leq \Phi_{n+1}$, $\Phi_n = 0$ on a neighborhood of N (which depends on n) and $\Phi_n(x) \uparrow +\infty$ if $x \in N^c$. This is possible since N is closed. If we let

$$g_n(x) = E_x[\exp(-\Phi_n(\xi(\tau_\Delta)))]$$

$$I_n(x) = -\log[g_n(x)],$$

then $g_n(x) \downarrow g(x)$ and $I_n(x) \uparrow I(x)$. It is sufficient therefore to prove that for each fixed n,

$$E[\int_0^{\tau_\Delta} L(\eta^v(s), v(s))ds] \geq I_n(x_0).$$

Just as before, g_n satisfies (1.7) and I_n satisfies (1.11). Now however, $I_n \in C(\bar{\Delta})$ with $I_n(x) = \Phi_n(x)$ on $\partial\Delta$. The standard application of Itô's lemma to equation (1.11) for $I_n(x)$ implies that for any subdomain D with $\bar{D} \subseteq \Delta$,

$$I_n(x_0) = E[\int_0^{\tau_D \wedge T} L(\eta^v(s), v(s))ds + I_n(\eta^v(\tau_D \wedge T))].$$

Now, letting $D \uparrow \Delta$ and $T \uparrow +\infty$ works. $\Phi_n(x) = 0$ on a neighborhood of N, and the admissibility condition implies that $\eta(\tau_D \wedge T)$ is in this neighborhood as $D \uparrow \Delta$ and $T \uparrow +\infty$ with probability one. Since $I_n(x)$ is a bounded function, the dominated convergence theorem implies that

$$E[I_n(\eta^v(\tau_D \wedge T))] \to 0 \quad \text{as} \quad D \uparrow \Delta \quad \text{and} \quad T \uparrow +\infty.$$

The convergence of the L term is the same as before and we conclude that

$$I_n(x_0) \leq E[\int_0^{\tau_\Delta} L(\eta^v(s),v(s))ds].$$

Letting $n \uparrow \infty$ completes the proof of the theorem.

III. A SIMPLE EXAMPLE: THE BESSEL PROCESS

The theorem above implies that whenever we can explicitly solve the Dirichlet problem for the generator of $\xi(t)$ on Δ with boundary values X_N, we can give an explicit solution for an associated stochastic control problem. As an example, let $d = 1$ and $\Delta = [0,a]$ (for any $a > 0$). If $b(x) \equiv 0$ and $\sigma(x) \equiv 1$ then $\xi(t)$ is simply Brownian motion. Let $N = \{a\}$. We have then

$$g(x) = P_x[\xi(\tau_\Delta) = a] = \frac{x}{a},$$

$$I(x) = -\log(\frac{x}{a}), \quad \nabla I(x) = \frac{-1}{x},$$

$$v^*(x) = 0 - (\frac{-1}{x}) = \frac{1}{x},$$

$$d\eta^*(t) = \frac{1}{\eta^*(t)}\, dt + dw.$$

Thus $\eta^*(t)$ is the Bessel process associated with 3-dimensional Brownian motion $\beta(t)$; $\eta^*(t) = |\beta(t)|$, see [4]. In the present case $L(x,v) = \frac{1}{2} v^2$. According to our theorem, the Bessel process $\eta^*(t)$ minimizes

$$E_x\left[\int_0^{\tau_\Delta} \frac{1}{2}\, v(\eta(s))^2 ds\right]$$

among all $d\eta = v(t)dt + dw$ for which $P[\eta(\tau_\Delta) = 0] = 0$.

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